Recurrence Relations for Orthogonal Polynomials and Algebraicity of Solutions of the Dirichlet Problem

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Dedicated to Professor Vladimir Maz'ya in recognition of his substantial contribution to the subject of Sobolev spaces

Abstract We show that any finite-term recurrence relation for planar orthogonal polynomials in a domain implies that the domain must be an ellipse. Our proof relies on Schwarz function techniques and on elementary properties of functions in Sobolev spaces.

1 Introduction

Let Ω be a bounded simply connected domain in the complex plane, and let $\{p_n\}_{n=0}^{\infty}$ denote the sequence of *Bergman orthogonal polynomials* of Ω,

$$p_n(z) = \gamma_n z^n + \gamma_{n-1} z^{n-1} + \cdots, \quad \gamma_n > 0, \quad n = 0, 1, 2, \dots,$$

of polynomials which are orthonormal with respect to the inner product

$$\langle f, g \rangle := \int_\Omega f(z)\overline{g(z)} dA(z),$$

where dA is the area measure. The associated L^2-norm is defined by

$$\|f\|_{L^2(\Omega)} := \langle f, f \rangle^{\frac{1}{2}} = \left\{ \int_\Omega |f(z)|^2 dA(z) \right\}^{\frac{1}{2}}.$$

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Let $\Omega' := \overline{\mathbb{C}} \setminus \overline{\Omega}$ denote the complement (in $\overline{\mathbb{C}}$) of $\overline{\Omega}$, and let Φ denote the conformal map $\Omega' \to \mathbb{D}' := \{w : |w| > 1\}$ normalized so that, near infinity,

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots, \quad \gamma > 0. \tag{1.1}$$

Note that the constant $1/\gamma$ gives the (logarithmic) *capacity* $\operatorname{cap}(\Gamma)$ of the boundary Γ of Ω (cf., for example, [19, 20]). The inverse conformal map $\Psi := \Phi^{-1} : \mathbb{D}' \to \Omega'$ has a Laurent series expansion of the form

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots, \tag{1.2}$$

valid for $|w| > 1$, where $b = 1/\gamma = \operatorname{cap}(\Gamma)$.

It is well known that orthogonal polynomials with respect to any measure μ on the real line do satisfy a three-term recurrence relation (cf., for example, [20]). By contrast, polynomials orthogonal with respect to the area measure, or the arc-length measure, in the complex plane $\mathbb{C}$, do not favor recurrence relations. To this end, Lempert [12] produced examples of several, rather special domains, where the associated orthogonal polynomials do not satisfy ANY finite-term recurrence relation. Putinar and the second author noted [14] that the fact that "the Bergman polynomials of Ω satisfy a finite-term recurrence relation" is, actually, equivalent to the fact that "any Dirichlet problem in Ω, with polynomial data, possesses a polynomial solution." The latter is the hypothesis of the so-called Khavinson–Shapiro conjecture [11] which states that *only ellipses* (*or ellipsoids in higher dimensions*) *have this property.* This conjecture has attracted some attention and the reader is referred to [2, 15, 8, 10] and the references therein for results reporting on the recent progress in that direction. In [14], the authors showed that if the Bergman polynomials of a simply connected domain Ω satisfy *any* finite-term recurrence relation and, in addition, the (necessarily algebraic) boundary of Ω, $\partial\Omega \subseteq \{P(x, y) = 0, P$ is a polynomial$\}$ satisfies the condition

(B) the set $\{P = 0\}$ is bounded in $\mathbb{C}$,

then Ω is an ellipse and the recurrence relation must be a three-term relation.

The main point of this paper is to remove the assumption (B). We do this, however, by assuming a finite-term recurrence of constant width, rather than one of variable width, as was the case in [14]. More precisely, we show that if the Bergman orthogonal polynomials of Ω satisfy an $(N+1)$-term recurrence relation, with N a positive integer, then Ω is an ellipse and $N = 2$.

Yet, in order for our argument to work, it is not enough to assume that Ω is merely simply connected, though a C^2-smooth Jordan boundary curve is sufficient. It remains an open question whether our results hold for any simply connected domain. We strongly believe so, but we were not able to extend our proof to that case.

2 The Main Results

Let Ω be a bounded simply connected planar domain. Consider the Bergman space $L_a^2(\Omega)$ associated with Ω. It is the Hilbert space of functions analytic and square integrable in Ω. In this paper, we assume that the boundary Γ of Ω is a Jordan curve. Under this assumption, the Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of Ω form a complete orthonormal system in $L_a^2(\Omega)$ (cf., for example, [9] for weaker assumptions on Γ regarding completeness in $L_a^2(\Omega)$).

A standard way to construct the Bergman polynomials is by means of the Gram–Schmidt process. This is a linear algorithm that computes the sequence of the orthonormal polynomials recursively, by using as data the entrances of the complex moment matrix $H := [\mu_{m,n}]_{m,n=0}^{\infty}$ of Ω:

$$\mu_{m,n} := \int_{\Omega} z^m \overline{z}^n dA(z). \tag{2.1}$$

It turns out that the complex moment matrix H alone suffices to determine the (unique) sequence of Bergman polynomials of Ω, and this determination is done, for each p_n, in a finite number of steps and by using a finite section of the moment matrix. (For more details regarding the general question of uniqueness properties of complex moments see [5].)

It is clear that for all $n = 0, 1, 2, \ldots$

$$zp_n(z) = \sum_{k=0}^{n+1} a_{k,n} p_k(z), \quad n = 0, 1, \ldots, \tag{2.2}$$

where the Fourier coefficients $a_{k,n}$ are given by $a_{k,n} = \langle zp_n, p_k \rangle$. Then

$$\sum_{k=0}^{n+1} |a_{k,n}|^2 < \infty, \quad n = 0, 1, \ldots .$$

The coefficients $a_{k,n}$ constitute the entrances of an infinite *upper Hessenberg matrix* M. This matrix is closely related to the multiplication operator by z (the *Bergman shift* operator) $T_z : L_a^2(\Omega) \to L_a^2(\Omega)$, defined by $(T_z f)(z) = zf(z)$, in the sense that T_z can be represented with respect to the basis $\{p_n\}_{n=0}^{\infty}$ by M. Note that T_z is linear and bounded on $L_a^2(\Omega)$.

Definition 2.1. We say that the Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy an $(N+1)$-*term recurrence relation* for some fixed positive integer N if for any $n \geqslant N-1$,

$$zp_n(z) = a_{n+1,n} p_{n+1}(z) + a_{n,n} p_n(z) + \ldots + a_{n-N+1,n} p_{n-N+1}(z). \tag{2.3}$$

If the Bergman polynomials satisfy an $(N+1)$-term recurrence relation then one easily sees (cf. [14]) that the adjoint operator T_z^* of the Bergman shift, and its multiples, increase the degree of a polynomial $p(z)$ subject to the constraint

$$\deg[(T_z^*)^m p] \leqslant m(N-1) + \deg p, \quad m \in \mathbb{N}. \tag{2.4}$$

This follows easily from the fact that T_z^* can be represented with respect to the basis $\{p_n\}_{n=0}^{\infty}$ by the adjoint matrix M^* of M which, in this case, has a lower Hessenberg and banded form of constant width $N+1$.

The next result confirms the Khavinson–Shapiro conjecture (cf. [11, 2, 15]) under an additional assumption on the degree of the polynomial solution to the Dirichlet problem.

Theorem 2.1. *Let Ω be a bounded simply connected domain in $\mathbb{C}$ with a C^2-smooth Jordan boundary Γ. Assume that there exists a positive integer $N := N(\Omega)$ with the property that the Dirichlet problem*

$$\Delta u = 0 \quad in \quad \Omega, \quad u = \overline{z}^m z^n \quad on \quad \Gamma, \tag{2.5}$$

has a polynomial solution of analytic degree $\leqslant m(N-1)+n$ (in z) and of conjugate analytic degree $\leqslant n(N-1)+m$ (in $\overline{z}$), for all positive integers m and n. Then Ω is an ellipse and $N = 2$.

Remark 2.1. Considering the polynomial $p(x,y) = \overline{z}z (= x^2 + y^2)$ in (2.5), it is easy to see that, under the assumptions of Theorem 2.1, the boundary curve Γ must be a part of the zero set of an algebraic polynomial and hence a piecewise analytic curve.

It is well known that the Bergman polynomials of an ellipse satisfy a three-term recurrence relation. In fact, it is easy to check that they are suitably normalized Chebyshev polynomials of the 2nd kind. The associated Hessenberg matrix in this case is triangular and goes by the name of *Jacobi matrix*. The following theorem states that this is the only possible case for an $(N+1)$-recurrence to occur.

Theorem 2.2. *With Ω and Γ as in Theorem* 2.1*, assume that the Bergman orthogonal polynomials for Ω satisfy an $(N+1)$-term recurrence relation, with some $N \geqslant 2$. Then Ω is an ellipse and hence $N = 2$.*

We note that the conclusion of Theorem 2.2 for polynomials orthogonal with respect to the arc-length measure that satisfy a three-term recurrence relation (i.e., under the assumption $N = 2$) goes back to Duren [7]. A similar result, as that of [7], but for polynomials orthogonal with respect to the harmonic measure on Γ, was established in [6].

Theorem 2.1 becomes an easy consequence of Theorem 2.2, after we establish the equivalence between the assumptions of the two theorems. This latter task was essentially done in [14, Theorem 1] under a more general definition for recurrences and thus without specific reference to the degree of the polynomial solution of (2.5). For our purposes here, however, we require the following explicit version of Theorem 1 of [14].

Proposition 2.1. *Let Ω be a bounded simply connected domain in $\mathbb{C}$ with a C^2-smooth Jordan boundary Γ. Then there exists a positive integer $N := N(\Omega)$ such that for all positive integers m and n the Dirichlet problem* (2.5) *with polynomial data $\overline{z}^m z^n$ has a polynomial solution of degree $\leqslant m(N-1)+n$ in z and $\leqslant n(N-1)+m$ in $\overline{z}$ if and only if the Bergman orthogonal polynomials for Ω satisfy an $(N+1)$-term recurrence relation.*

The following result, which gives the ratio asymptotics for the Bergman polynomials, is needed in establishing Theorem 2.2. Its proof is a simple consequence of the strong asymptotics for Bergman polynomials over domains with smooth boundaries, established by Suetin [19, Theorem 1.2] and, thus, we omit it.

Lemma 2.1. *Assume that Ω is a bounded simply connected domain in $\mathbb{C}$ with a C^2-smooth Jordan boundary Γ. Let $\{p_n\}_{n=0}^{\infty}$ denote the sequence of Bergman polynomials of Ω. Then*

$$\lim_{n\to\infty} \frac{p_{n+1}(z)}{p_n(z)} = \Phi(z), \quad z \in \overline{\Omega'}. \tag{2.6}$$

We note, in passing, that strong asymptotics for Bergman polynomials were first derived by Carleman [3] under the assumption that Γ is analytic.

Remark 2.2. For Ω simply connected and bounded, a well-known result by Fejér asserts that the zeros of $p_n(z)$ ($n \in \mathbb{N}$) are contained in $Co(\overline{\Omega})$, where $Co(\overline{\Omega})$ denotes the convex hull of $\overline{\Omega}$. Under the additional assumption on Γ in Lemma 2.1, from [19, Theorem 1.2] it follows that there exists a positive integer n_0 such that the sequence $\{p_n\}_{n=n_0}^{\infty}$ has no zeros in Ω'.

Remark 2.3. Lemma 2.1 is precisely the reason we need to assume the C^2-smoothness of Γ in Theorem 2.1. Although we were not able to extend the ratio asymptotics to more general sets, we believe that (2.6) holds for arbitrary domains Ω, such that $\Gamma = \partial\Omega = \partial\Omega'$ is a continuum.

3 Proofs

Proof of Proposition 2.1. Fix two positive integers m, n and assume that the Bergman orthogonal polynomials for Ω satisfy an $(N+1)$-term recurrence relation. Then, in view of (2.4),

$$(T_z^*)^m z^{n-1} = q(z),$$

where q is a polynomial of degree $\leqslant m(N-1)+n-1$. Therefore,

$$\overline{z}^m z^{n-1} = q(z) + h(z), \quad z \in \Omega,$$

where $h \in L^2(\Omega) \ominus L_a^2(\Omega)$. Let $Q(z)$ be a polynomial satisfying $Q' = q$. According to the so-called Khavin lemma (cf., for example, [18, p. 26]), $h = \partial g$ with g in the Sobolev space $W_0^{1,2}(\Omega)$. Integrating against ∂, we find

$$\overline{z}^m z^n = Q(z) + g(z) + \overline{f(z)}, \quad z \in \Omega, \tag{3.1}$$

where $f \in L_a^2(\Omega)$. Since Γ is smooth, it follows that $g = 0$, a.e. on $\Gamma = \partial\Omega$ and thus

$$\overline{z}^m z^n = Q(z) + \overline{f(z)} \quad \text{for a. e. } \ z \in \Gamma.$$

Moreover, from (3.1), the Poincaré inequality and the smoothness of Γ, we infer easily (cf., for example, [1]) that f, in fact, belongs to the Hardy space $H^2(\Omega)$. (For the most up to date theory of Sobolev spaces we refer the reader to [13].) Similarly, we have

$$\overline{z}^n z^m = G(z) + \overline{f_1(z)} \quad \text{for a. e. } \ z \in \Gamma,$$

where G is a polynomial of degree $\leqslant n(N-1)+m$ and $f_1 \in L_a^2(\Omega) \cap H^2(\Omega)$. Hence $Q(z)+\overline{f(z)} = \overline{G(z)}+f_1(z)$, $\quad z \in \Omega$ and, since Ω is simply connected, we infer that $Q(z) = f_1(z)+c$ and $G(z) = f(z)+\overline{c}$, $z \in \Omega$, for some constant c. Hence the Dirichlet problem in Ω with data $\overline{z}^m z^n$ has a polynomial solution whose analytic degree (in z) is $\leqslant m(N-1)+n$ and its antianalytic degree (in $\overline{z}$) is $\leqslant n(N-1)+m$.

For the converse, assume that the Dirichlet problem for Ω with data $\overline{z}^m z^n$ has a polynomial solution $u(z) = Q(z) + \overline{G(z)}$, where Q and G are complex polynomials with $\deg(Q) \leqslant m(N-1)+n$.

Let $h(z)$ be a bounded analytic function in $\overline{\Omega}$. Then, by the Stokes and Cauchy theorems, for $n \geqslant 1$ we obtain

$$\begin{aligned}\langle (T_z^*)^m z^{n-1}, h\rangle &= \langle \overline{z}^m z^{n-1}, h\rangle = \int_\Omega \overline{z}^m z^{n-1}\overline{h(z)}\, dA(z)\\ &= -\frac{1}{2ni}\int_\Gamma \overline{\zeta^m}\zeta^n\overline{h(\zeta)}\, d\overline{\zeta} = \frac{1}{n}\int_\Omega Q'(z)\overline{h(z)}\, dA(z) = \langle q, h\rangle,\end{aligned}$$

where $q(z) := Q'(z)/n$. This implies

$$(T_z^*)^m z^{n-1} = q(z),$$

where $\deg(q) \leqslant m(N-1)+n-1$, and hence the finite-term recurrence relation for the Bergman polynomials. □

Proof of Theorem 2.2. Assume that the Bergman polynomials of Ω satisfy the recurrence relation (2.3) for some $N \geqslant 2$. Then from Proposition 2.1 and Remark 2.1 we see that Γ must be piecewise analytic.

Now we argue as in [7, p. 314]. For the moment, we assume that each of the $N+1$ sequences of the Fourier coefficients

$$\alpha_n^{(1)} := a_{n+1,n}, \quad \alpha_n^{(2)} := a_{n,n}, \quad \ldots, \quad \alpha_n^{(N+1)} := a_{n-N+1,n},\ n \in \mathbb{N}, \tag{3.2}$$

is bounded, and then proceed as follows:

(i) divide both sides of (2.3) by $p_n(z)$ (for $z \in \mathbb{C} \setminus Co(\overline{\Omega})$);

(ii) take the limit as $n \to \infty$, $n \in \Lambda$, on both sides of the resulting equation, where Λ is an appropriate subsequence of $\mathbb{N}$ chosen so that each sequence in (3.2) tends to a finite limit;

(iii) note that

$$\frac{p_{n-k}}{p_n} = \frac{p_{n-1}}{p_n}\frac{p_{n-2}}{p_{n-1}}\cdots\frac{p_{n-k}}{p_{n-k+1}}, \quad k \leqslant N-1;$$

(iv) apply Lemma 2.1.

The above steps yield that the inverse exterior conformal map $\Psi : \mathbb{D}' \to \Omega'$ has a finite Laurent expansion of the form

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots + \frac{b_{N-1}}{w^{N-1}}, \quad |w| > 1. \tag{3.3}$$

To verify that all the sequences in (3.2) are bounded, one simply has to apply the Cauchy–Schwarz inequality, for $j = 1, 2, \ldots, N+1$ and $n \geqslant N-1$:

$$\begin{aligned} |\alpha_n^{(j)}| = |a_{n+2-j,n}| &= \left| \int_\Omega \overline{z} p_{n+2-j}(z)\overline{p_n(z)}dA(z) \right| \\ &\leqslant \|z\|_\infty \ \|p_{n+2-j}\|_{L^2(\Omega)} \ \|p_n\|_{L^2(\Omega)} = \|z\|_\infty, \end{aligned}$$

where $\|\cdot\|_\infty$ stands for the sup-norm on $\overline{\Omega}$.

From (3.3) it follows that Ψ is a rational function. This implies that Ω' is an unbounded quadrature domain. Hence the associated Schwarz function $S(z)$ with $S(z) = \overline{z}$ on $\Gamma := \partial\Omega = \partial\Omega'$ has a meromorphic extension to Ω', i.e.,

$$S(z) = r(z) + \sum_{j=1}^{M}\sum_{l=1}^{k_j} \frac{c_{j,l}}{(z-z_j)^l} + f(z), \tag{3.4}$$

where $z_j \in \Omega' \setminus \{\infty\}$, $k_j \in \mathbb{N}$, $r(z)$ is a polynomial of degree d, and $f(z)$ is analytic and bounded in Ω' (cf., for example, [17]).

We first show that all the constants $c_{j,l}$, $j = 1, \ldots, M$, $l = 1, \ldots, k_j$, in (3.4) vanish.

Let $P(z) := \prod_{j=1}^{M}(z-z_j)^{k_j}$. Consider the Dirichlet problem (2.5) with data $\overline{z}P(z)$. Our hypothesis and Proposition 2.1 imply that there exist analytic polynomials $h(z)$ and $g(z)$ such that

$$\overline{z}P(z) = g(z) + \overline{h(z)}, \quad z \in \Gamma. \tag{3.5}$$

(Note that $\deg(h) \geqslant 1$; otherwise, on Γ, $\overline{z} = S(z)$ is equal to a rational function and $\Gamma = \partial\Omega$ is a circle, according to a well-known theorem of Davis [4, p. 104].) Let $R(z) = \overline{S(z)}$ be the anticonformal reflection about Γ. It is obvious that, by (3.4), $R(z)$ extends to Ω' and has poles at ∞ and $\{z_j\}_{j=1}^M$. From (3.4) and (3.5) we see that, on Γ,

$$g(z) + \overline{h(R(z))} = r(z)\,P(z) + F(z), \tag{3.6}$$

where F(z) is analytic in $\Omega' \setminus \{\infty\}$ and it may have a pole of order at most $\sum_{j=1}^{M} k_j$ at ∞. Since both sides of (3.6) are analytic functions of z, (3.6) holds on any path originating on Γ along which $S(z)$ continues analytically.

Now, let γ be any path in $\Omega' \setminus \{\infty\}$ joining Γ to a given pole z_j and avoiding all other poles. Then the right-hand side of (3.6) stays bounded on γ and so does $g(z)$, while $\overline{h(R(z))} \to \infty$ at z_j because $|R(z)| \to \infty$ at z_j and $h(z)$ is a (nonconstant) polynomial. This is a contradiction and therefore $S(z)$ can have no finite poles in Ω', i.e.,

$$S(z) = r(z) + f(z), \quad z \in \Omega', \tag{3.7}$$

where $f(z)$ is analytic in Ω' (including ∞).

Consider now the Dirichlet problem (2.5) with data $z\overline{z} = |z|^2$. In view of our hypothesis and Proposition 2.1, we can find a polynomial $g(z)$ of degree $k \geqslant 2$ (since if $k \leqslant 1$, then Γ is obviously a circle and $N = 1$) such that for $z \in \Gamma$

$$2\mathrm{Re}\{g(z)\} = g(z) + \overline{g(z)} = |z|^2 = zS(z) \tag{3.8}$$

or, by (3.7),

$$g(z) + \overline{g(R(z))} = z\,r(z) + zf(z), \quad z \in \Gamma. \tag{3.9}$$

Consider a path γ in $\Omega' \setminus \{\infty\}$ joining Γ to ∞. Since both sides of (3.9) are analytic in Ω', (3.9) holds along γ. Yet, near ∞ we have $|g(z)| \sim |z|^k$, $|\overline{g(R(z))}| \sim |z|^{dk}$, and the right-hand side of (3.9) behaves as $O(|z|^{d+1})$, This can only be possible if $dk = d + 1$, i.e., since $k \geqslant 2$, only if $d = 1$ and $k = 2$. From this it follows that Ω is an ellipse and $N = 2$. However, it may be worthwhile to point out the following observation as well. Thus,

$$S(z) = c\,z + f(z), \quad z \in \Omega', \tag{3.10}$$

with f analytic in Ω'.

But this implies right away that Ω' is a null-quadrature domain (cf. [17] and [18]). Indeed, using the Green and Cauchy theorems, for any number $m \geqslant 3$ we have

$$\begin{aligned}\int_{\Omega'} \frac{1}{z^m} dA(z) &= -\frac{1}{2i}\int_{\Gamma} \frac{1}{z^m}\overline{z}dz = -\frac{1}{2i}\int_{\Gamma} \frac{1}{z^m} S(z)dz \\ &= -\frac{1}{2i}\int_{|z|=\rho} \frac{1}{z^m}\{c\,z + f(z)\}dz = 0\end{aligned}$$

for large enough ρ since f is analytic in Ω' (including ∞). From this, in view of [16, Theorem 1], we infer that Ω' must be the exterior of an ellipse. Hence Ω is an ellipse and thus $N = 2$. □

4 Concluding Remarks

We finish with a number of remarks.

(i) As we have pointed out above, the main place where the C^2-regularity of the boundary $\Gamma = \partial\Omega$ is needed was the application of the strong asymptotics for Bergman polynomials of Suetin [19], that yield Lemma 2.1. Moreover, it is clear from the proof of Theorem 2.2 that we only need (2.6) to hold on a continuum subset of Ω', in a neighborhood of ∞. It looks quite plausible that, in this weaker form, (2.6) holds for arbitrary, bounded Jordan domains. Yet, we have not been able to derive it for such general domains or find a pertinent result in the literature.

(ii) For the most updated account on the status of the Khavinson–Shapiro conjecture in its full generality, mostly due to the work of Render [15], we refer the reader to the recent survey [10].

(iii) For a quite different approach, regarding singularities of solutions of the Dirichlet problem in $\mathbb{R}^2$, we refer the reader to [2] (cf. also [8]).

(iv) The finiteness of only the first column of the adjoint matrix M^* associated with T_z^* is not sufficient to yield Theorems 2.1–2.2 or Proposition 2.1 (unless, of course, $a_{0,n} = 0$, $n \geqslant 2$; cf. [14, Prop. 1]). For example, take Γ to be the bounded component of $\{x^2 + y^2 - 1 + \epsilon(x^3 - 3xy^2) = 0\}$, where $\epsilon > 0$ is small enough so that Γ is a perturbation of the unit circle. Then the quadratic data $z\overline{z}$ are matched on Γ by a cubic harmonic polynomial, despite the fact that Γ is not an ellipse.

(v) The assumption that Ω is simply connected is not really necessary. As is seen from the arguments of [2], the hypothesis in the Khavinson–Shapiro conjecture implies that Ω is simply connected.

(vi) The condition that a finite-term recurrence relation (of some constant width $N+1$), satisfied by the Bergman polynomials of Ω, is stronger than the hypothesis of the Khavinson–Shapiro conjecture for Ω. This is so because the Khavinson–Shapiro conjecture does not involve any assumption on the degree of the polynomial solution. *Thus, a full proof of the conjecture is still amiss.*

(vii) If the hypothesis of the Khavinson–Shapiro conjecture is satisfied then, clearly, $\Gamma = \partial\Omega$ is algebraic and hence piecewise analytic. Yet, in order to be able to use the ratio asymptotics for the Bergman polynomials, as they have been obtained by Suetin [19], we must eliminate the possibility that Γ has cusps. Perhaps, whenever the hypothesis of the Khavinson–Shapiro conjecture holds (cf. Proposition 2.1) the cusps cannot occur a priori. We were not able to prove this either. We note however that it is possible to have cusped curves on which a quadratic matches a harmonic polynomial, for example, $y^2 = x^3 - 3y^2x$.

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